Signal Classification Using a Peak-to-Average Power Ratio Statistic

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***Abstract*—This paper addresses signal classification based on an average power statistic for peak-to-average-power ratio (PAR) reduced signals. Specifically, it is assumed that either a QAM or a Complex Gaussian finite-length symbol is transmitted in a noisy, peak-limited channel with known peak power. The goal is determine the whether an uninformed receiver can distinguish between these signal types using an average power statistic. Several methods for transmitting through peak-power channels are examined including optimal clipping and piecewise linear scaling (PWLS) with selected mapping (SLM) PAR reduction. For the analysis, it is necessary to derive the mean power for each of the transmission methods. Accordingly, we show how the harmonic mean PAR, E[1/PAR], is related to the mean power and derive E[1/PAR] in closed form. We find that average power is a accurate discriminator for low-order QAM and Gaussian symbols. For high-order QAM, accurate discrimination is also possible when the noise level is sufficiently low or when enough signal samples are available.**

I. Introduction

In wireless communications applications there are certain situations when it is desirable to determine the mean transmit power of a signal. For example, knowing the mean transmit power can be useful for ranging. Or, if the channel is known, a mean transmit power estimate may help distinguish the transmitter's equipment, which could be useful for system identification purposes. Finally, if both the channel and the maximum transmit power are known, mean transmit power can be used to discriminate between modulation types. Many different approaches have been proposed for modulation classification, see, for example, [1], [2], and references therein for more information. With this in mind, the following work is a theoretical characterization of the mean transmit power, given the maximum transmit power, for several different operating scenarios. As most physical transmit power amplifiers (PAs) are peak-power limited, it is important to assess the average transmit power under a peak-transmit power constraint.

The modulations examined include Gaussian signals, which are well approximated with orthogonal frequency division multiplexing (OFDM) waveforms [3], and QAM signals which are commonly seen in single carrier modulation formats. These are popular modulation formats, which can both be found as options in the 802.16 WiMax standard [4]. In this paper the Nyquist-sampled signals are used for analysis. Furthermore, it is assumed throughout that the PA channel takes on a soft limiter characteristic with a linear phase response and a peak power of P_{max}, i.e.

$$g(x) = \begin{cases} \sqrt{P_{max}} x, & |x| \leq 1 \\ \sqrt{P_{max}}\, e^{j\angle x}, & |x| > 1, \end{cases} \quad (1)$$

which is a reasonable assumption when a linear PA or when PA pre-distortion is implemented. For peak-limited channels, many performance enhancing schemes have been proposed for signals that have a large dynamic range, such as OFDM. A broad class of these techniques involve reducing the signal peak-to-average-power ratio (PAR) [3], which is defined as

$$PAR[\mathbf{x}] = \frac{\|\mathbf{x}\|_\infty^2}{E[\|\mathbf{x}\|_2^2]/N}, \quad (2)$$

where $\mathbf{x}$ is a length-N vector that represents one transmit symbol. The analysis here is constrained to random-search PAR reduction methods like selected mapping (SLM) [5] or partial transmit sequence (PTS) [6], which generate multiple representations of the signal to be transmitted with a random power-conserving transformations and select the version with the lowest PAR. For SLM, [7] shows that the mappings are approximately statistically independent in the time domain under certain conditions on the frequency domain phase rotation sequences. In a generalized random-search scheme, the length-N symbol to be transmitted, $\mathbf{x}$, is modified to create D different representations of the original signal, $\mathbf{x}^{(d)}, d \in \{1, 2, ..., D\}$. Then the transmitted signal is the minimum PAR sequence, $\mathbf{x}^{(\tilde{d})}$, where

$$\tilde{d} = \arg \min_{d\in\{1,2,\ldots,D\}} PAR[\mathbf{x}^{(d)}]. \quad (3)$$

In transmitting through a peak-limited channel, the signal may experience distortion effects from clipping. In [8], the optimal scaling factor was derived to maximize the signal-to-noise-plus-distortion ratio (SNDR) for a channel with a given peak-limit. However, if zero distortion is desired, it is possible to use the piece-wise linear scaling (PWLS) method to avoid all distortion [9]. SNDR-optimal variations on PWLS that include clipping are discussed in [10].

The assumption here is that either PWLS or optimal clipping is used for Gaussian signaling (e.g. OFDM) or that PWLS is used for QAM signaling. With this assumption, we determine theoretically the ratio of mean transmit power to maximum transmit power for each scenario. Finally, we outline the mean-power based discriminator that can be used to distinguish between QAM signals and Gaussian signals, given that they are both transmitted with the same peak power. In addition to this result, we also derive the closed-form expression for the harmonic mean of the PAR for SLM OFDM, which was been acknowledged as an important quantity in [9] and which we

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show plays an important role in determining the mean transmit power in PWLS OFDM systems.

II. Mean Power Derivation

A. PWLS

In a PWLS system the symbol is scaled by its maximum power such that

$$\mathbf{x}_{pwls} = \frac{\sqrt{P_{max}}}{\|\mathbf{x}\|_\infty}\mathbf{x}. \tag{4}$$

The average power can then be calulated by

$$\frac{E[\|\mathbf{x}_{pwls}\|_2^2]}{N} = \frac{P_{max}}{N} E\left[\frac{\|\mathbf{x}\|_2^2}{\|\mathbf{x}\|_\infty^2}\right]. \tag{5}$$

An intuitive approximation for the mean power of a PWLS symbol can be made where

$$\frac{P_{max}}{N} E\left[\frac{\|\mathbf{x}\|_2^2}{\|\mathbf{x}\|_\infty^2}\right] \approx \frac{P_{max}}{N} E\left[\frac{1}{\|\mathbf{x}\|_\infty^2}\right] E\left[\|\mathbf{x}\|_2^2\right] \tag{6}$$

$$= P_{max} E\left[\frac{1}{PAR}\right]. \tag{7}$$

This approximation follows by assuming that $\|\mathbf{x}\|_2^2$ and $\|\mathbf{x}\|_\infty^2$ are approximately independent, which is reasonable for large N because $\|\mathbf{x}\|_\infty^2$ and $\|\mathbf{x}\|_2^2$ only share one common term for each realization of $\mathbf{x}$. However, for small N, it is clear that there may be significant interdependence between $\|\mathbf{x}\|_2^2$ and $\|\mathbf{x}\|_\infty^2$.

1) Gaussian Symobls: Note that a lower bound on $E[1/PAR]$ for SLM PAR reduction (with $D = 1$ meaning that no PAR reduction occurs) can be derived from the work in [11] using Jensen's inequality according to

$$E[1/PAR] \geq \frac{1}{E[PAR]} \tag{8}$$

$$= \left(\frac{(D-1)!}{N} \sum_{k=1}^{N} \prod_{p=0}^{D-1} \frac{1}{(\frac{k}{N} + p)}\right)^{-1}. \tag{9}$$

Here we extend this result by deriving $E[1/PAR]$. Using the the pdf of the PAR [11]

$$f_{PAR}(x) = NDe^{-x}(1-(1-e^{-x})^N)^{D-1}(1-e^{-x})^{N-1}, \tag{10}$$

the mean power can be caluclated as

$$E[1/PAR] = \int_0^\infty 1/x \cdot f_{PAR}(x) dx. \tag{11}$$

For the PAR pdf in (10), we find that

$$E[1/PAR] = ND \sum_{n=0}^{N-1} \sum_{d=0}^{D-1} \sum_{p=0}^{Nd} (-1)^{n+d+p+1} \binom{N-1}{n}\binom{D-1}{d}\binom{Nd}{p} \ln(n+p+1). \tag{12}$$

A detailed proof of this derivation is provided in the appendix. When $D = 1$, and no SLM PAR reduction is performed, (12) reduces to

$$E[1/PAR] = N \sum_{n=0}^{N-1} (-1)^{n+1} \binom{N-1}{n} \ln(n+1)$$

$$= N \ln(A_N), \tag{13}$$

where, by expanding $e^{E[1/PAR]/N}$ from (13) we can see that A_N is the Nth term of the series

$$A = \left\{\frac{2}{1}, \frac{2^2}{1\cdot 3}, \frac{2^3\cdot 4}{1\cdot 3^3}, \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}, \ldots\right\}. \tag{14}$$

Interestingly, efficient calculation of the terms of A has been explored because the terms of A are also terms in the infinite product expansion series of $\pi/2$, e and e^γ, where γ is the Euler-Mascheroni constant, see [12] for details.

In practice, communications signals that have a Gaussian nature like OFDM only have iid samples at the Nyquist sampling rate. Instead, the continuous time symbol is a continuous Gaussian process with a certain band limited power spectral density (PSD), i.e. correlated samples. Thus, the expression for $E[1/PAR]$ in (12) is an upper bound on the continuous-time value for $E[1/PAR]$.

2) QAM Symbols: Finding an exact value for the harmonic mean of an arbitrary QAM vector is surprisingly harder than deriving the complex Gaussian case. By writing the magnitude squared density function as a sum of Dirac delta functions with weights corresponding to the probability, we have

$$f_{|x|^2}^{(QAM)}(x) = \sum_{k=1}^{N} p_k \delta(x - m_k). \tag{15}$$

For example, a 16QAM signal with a mean power of one has signal power distributed according to the pdf

$$f_{|x|^2}^{(16QAM)}(x) = \frac{1}{4}\left(\delta\left(x-0.2\right) + \delta\left(x-1.8\right)\right) + 1/2\delta\left(x-1\right). \tag{16}$$

Thus, assuming independent samples, the PAR can be expressed as

$$Pr(PAR > \gamma) = \left(1 - \sum_{k=1}^{N} p_k U(\gamma - m_k)\right)^N, \tag{17}$$

where $U(x)$ is the unit step function, which is an N-order multinomial that is hard to simplify further.

A useful upper bound on the PAR, PAR_{ub} can be found by assuming the maximum-power constellation point is part of the symbol. Here PAR_{ub} is defined such that

$$Pr(PAR > PAR_{ub}) = 0. \tag{18}$$

Thus, the upper bound on the PAR of MQAM distribution can be calculated by determining the power of this maximum point divided by the mean power. These values are computed in Table I.

For example, the worst-case QAM in a PAR sense is ∞QAM where both the real and imaginary parts of each element, $x \sim U[-c, c]$, are uniformly distributed. Thus the pdf of $|x|^2$ has support on $[0, 2c^2]$ and the mean power is $E[|x|^2] = 2c^2/3$. As $N \to \infty$, we have

$$Pr\left(PAR[\mathbf{x}] = 3\right) \to 1, \tag{19}$$

So, $E[PAR^{(QAM)}] \leq 3$ and $E[1/PAR^{(QAM)}] \geq \frac{1}{3}$.

TABLE I
PAR UPPER-BOUND ON MQAM.

M	$PAR_{ub}(dB)$	$1/PAR_{ub}$	$\frac{\sigma^2_{\|x\|^2}}{PAR^2_{ub}}$
4	0	1	0
16	2.6	0.56	0.057
64	3.7	0.43	0.062
256	4.2	0.38	0.059
∞	4.8	0.33	0.053

B. Optimal Clipping for Gaussian Symbols

In optimal clipping, some clipping distortion is allowed in order to achieve a higher transmit power. We assume that the optimal clipping is the same for all symbols and is dependent on the AWGN noise power. More precisely we assume that the received signal is $g(\eta\mathbf{x}) + \mathbf{w}$, where $\mathbf{w}$ is the AWGN channel noise. The goal is to determine the mean power, $E[\|g(\eta\mathbf{x})\|_2^2]/N$ when η is chosen so that the signal-to-noise-plus-distortion ratio (SNDR) is maximized. SNDR is defined by

$$SNDR = \frac{|\alpha|^2 \mathrm{E}[\mathbf{x}^{\mathcal{H}}\mathbf{x}]}{\mathrm{E}[\mathbf{d}^{\mathcal{H}}\mathbf{d}] + \mathrm{E}[\mathbf{w}^{\mathcal{H}}\mathbf{w}]}, \tag{20}$$

where $\mathbf{d}$ is the uncorrelated distortion noise and α is chosen so that $\mathrm{E}[\mathbf{x}^{\mathcal{H}}\mathbf{d}] = 0$.

In [8], the optimal η was determined for any arbitrary magnitude distribution on an iid $\mathbf{x}$. Define $\mathbf{r} = g(\eta\mathbf{x})$. Now, by inferring that $\alpha = \mathrm{E}[\mathbf{x}^{\mathcal{H}}\mathbf{r}]/(N\sigma_x^2)$ and that $\mathrm{E}[\mathbf{d}^{\mathcal{H}}\mathbf{d}] = \mathrm{E}[\mathbf{r}^{\mathcal{H}}\mathbf{r}] - |\mathrm{E}[\mathbf{x}^{\mathcal{H}}\mathbf{r}]|^2/(N\sigma_x^2)$, (20) can be rewritten as

$$SNDR = \frac{|\mathrm{E}[\mathbf{x}^{\mathcal{H}}\mathbf{r}]|^2}{\mathrm{E}[\mathbf{r}^{\mathcal{H}}\mathbf{r}]N\sigma_x^2 - |\mathrm{E}[\mathbf{x}^{\mathcal{H}}\mathbf{r}]|^2 + N^2\sigma_w^2\sigma_x^2}, \tag{21}$$

where σ_w^2 is the noise power. For Gaussian samples $\mathbf{x}$, the SNDR-optimal η can be calculated using $\eta = 1/\sigma_x q(P_{max}/\sigma_w^2)$, where $q(\cdot) = T^{-1}(\cdot)$, the inverse of

$$T(x) = \frac{2x}{\sqrt{\pi}\mathrm{erfc}(x)}. \tag{22}$$

Other cases are also treated in [8].

Once η is calculated, it is straightforward to show that the mean power can be calculated by

$$E[|g(\eta\mathbf{x})|^2] = P_{max}\Bigg(\int_0^1 \frac{x}{\eta^2} f_{|x|^2}\left(\frac{x}{\eta^2}\right)dx + \int_1^\infty \frac{1}{\eta^2} f_{|x|^2}\left(\frac{x}{\eta^2}\right)dx\Bigg). \tag{23}$$

and the second moment of the power is

$$E[|g(\eta\mathbf{x})|^4] = P_{max}^2\Bigg(\int_0^1 \frac{x^2}{\eta^2} f_{|x|^2}\left(\frac{x}{\eta^2}\right)dx + \int_1^\infty \frac{1}{\eta^2} f_{|x|^2}\left(\frac{x}{\eta^2}\right)dx\Bigg). \tag{24}$$

For the complex Gaussian case, this simplifies to

$$E[|g(\eta\mathbf{x})|^2] = P_{max}\frac{1 - e^{-\alpha}}{\alpha} \tag{25}$$

and the power variance is

$$\sigma^2_{|g(\eta\mathbf{x})|^2} = P_{max}^2 \frac{2e^{-\alpha}}{\alpha}\left(\frac{\sinh(\alpha)}{\alpha} - 1\right), \tag{26}$$

where $\alpha = q(P_{max}/\sigma_w^2)^2$ and $\sinh(x) = (e^x - e^{-x})/2$.

III. HYPOTHESIS-TEST DISCRIMINATION

The goal is to distinguish whether a received signal is complex Gaussian or QAM based on an average power estimate. Assuming independent AWGN noise, w and K signal samples (which may be more or less than N), the average power statistic is

$$S = \frac{1}{KP_{max}} \sum_{k=1}^{K} |r(x) + w|^2, \tag{27}$$

where $r(x)$ is either the PWLS operation in (4) or the optimal clipping operation $g(\eta x)$ discussed in §II.B. If K is large enough, S will be approximately Gaussian distributed, such that the statistic given a QAM signal is

$$S^{(Q)} \sim \mathcal{N}\left(\frac{1}{PAR_{ub}} + \tilde{\sigma}_w^2, \frac{\sigma^2_{|x|^2}}{KPAR^2_{ub}} + \frac{\tilde{\sigma}_w^4}{K}\right), \tag{28}$$

where $\tilde{\sigma}_w^2 = \sigma_w^2/P_{max}$ and $1/PAR_{ub}$ and $\frac{\sigma^2_{|x|^2}}{PAR^2_{ub}}$ are given in Table I. Assuming a complex Gaussian signal and optimal clipping the statistic is

$$S^{(G,oc)} \sim \mathcal{N}\left(\frac{E[|g(\eta x)|^2]}{P_{max}} + \tilde{\sigma}_w^2, \frac{\sigma^2_{|g(\eta\mathbf{x})|^2}}{KP^2_{max}} + \frac{\tilde{\sigma}_w^4}{K}\right), \tag{29}$$

or assuming PWLS and that the PWLS-scale samples are iid $x \sim \mathcal{CN}(0, P_{max}E[1/PAR])$, the statistic is

$$S^{(G,pwls)} \sim \mathcal{N}\left(E[1/PAR] + \tilde{\sigma}_w^2, \frac{E[1/PAR]^2}{K} + \frac{\tilde{\sigma}_w^4}{K}\right). \tag{30}$$

From [13, §6.2] we know that the likelihood ratio test minimizes the probability of error, which is defined as the probability of classifying a signal as having one distribution when it actually has another. To simplify the equations, we assume that the prior probabilities are equal and that type I and II errors are equally costly. With this, a threshold range can be found where, assuming that $\sigma_0 < \sigma_1$, the null hypothesis that $H_0 : S \sim \mathcal{N}(\mu_0, \sigma_0^2)$ is accepted if $S \in [\mu - \epsilon, \mu + \epsilon]$, where

$$\mu = \left(\frac{\mu_0}{\sigma_0^2} - \frac{\mu_1}{\sigma_1^2}\right)\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \tag{31}$$

and

$$\epsilon^2 = \left(\ln\frac{\sigma_1^2}{\sigma_0^2} + \frac{(\mu_1 - \mu_0)^2}{\sigma_1^2 - \sigma_0^2}\right)\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}. \tag{32}$$

If $S \notin [\mu - \epsilon, \mu + \epsilon]$, the alternative hypothesis that $H_1 : S \sim \mathcal{N}(\mu_1, \sigma_1^2)$ is accepted. The probability of incorrect classification is the probability mass of $S \sim \mathcal{N}(\mu_1, \sigma_1^2)$ outside the detection region, i.e.

$$P_e = 1 - \int_{\mu-\epsilon}^{\mu+\epsilon} f_{S(\mu_0,\sigma_0^2)}(x)dx, \tag{33}$$

which will decrease as the value of $E[1/PAR]$ or $E[|g(\eta x)|^2]$ decreases.

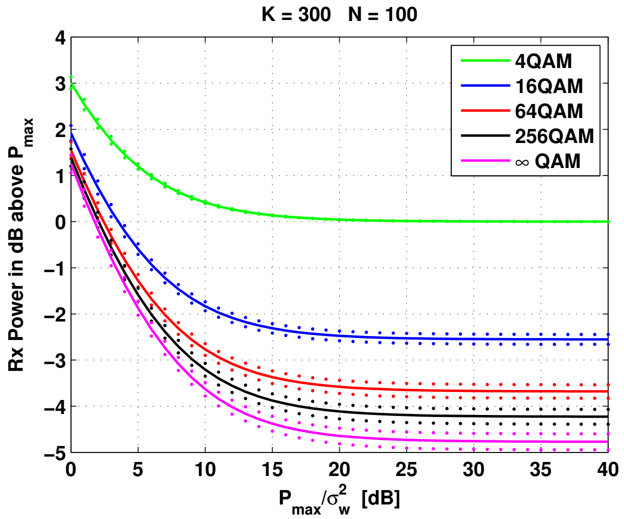

Fig. 1. Mean and $\pm\sigma$ power values for different QAM signals.

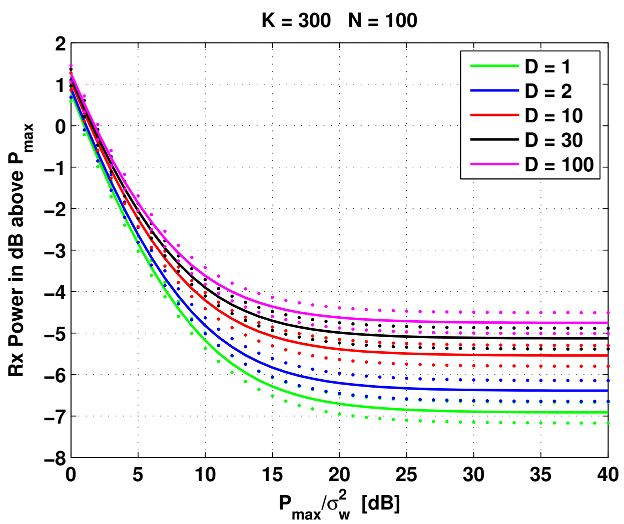

Fig. 2. Mean and $\pm\sigma$ power values for different PWLS signals with different values for the number of selected mappings, D.

IV. Simulations

First, to illustrate the effect of peak-limited mitigation methods on the mean power for the various signals we plot the mean power and the mean power plus or minus one standard deviation for QAM (Fig. 1), SLM PWLS (Fig. 2) and optimal clipping (Fig. 3). The plots show that for some parameter pairs the mean power is very similar between the complex Gaussian signals and the QAM signals. Such similarity may make mean power discrimination for those parameter pairs difficult. For instance, the mean power for an ∞QAM signal and a $D = 100$ PWLS signal are very similar. Also, it is clear from Fig. 3, that the number of samples, K, that contribute to the average power statistic significantly influence the standard deviation of the statistic, which will affect detection performance.

The effect of varying these parameters on the detection performance is summarized in Fig. 4 and 5. For these plots it is assumed that the noise power is known. The performance of detecting QAM versus an optimally clipped signal as shown in Fig. 4, is non-monotonic for higher-order QAM. This is because the mean power of QAM switches from being less than the clipped signal's mean power to being greater than it as P_{max}/σ_w^2 increases. The point at which the mean powers are the same maximizes the error rate curve at a probability of one half. By contrast, the PWLS/QAM detection has a more regular pattern in that all of the curves are decreasing in P_{max}/σ_w^2 up to a saturation point. This phenomenon is consistent with Fig. 2 and 3, which show that the mean power and the variance in QAM and PWLS level off to constant values. Thus, the probability of error in detecting one or the other also levels off.

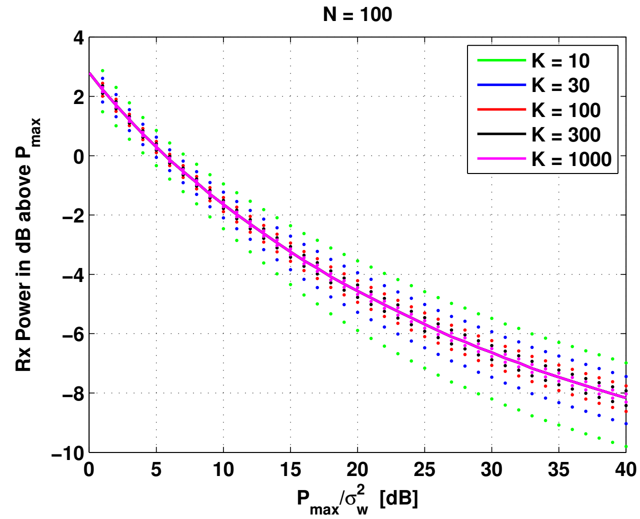

Fig. 3. Mean and $\pm\sigma$ power values for different optimally clipped complex Gaussian signals with different values for the number of samples, K, in the statistic.

The general picture from the simulations is that the detection performance heavily depends on the system parameters and noise level. It is clear that 4QAM can be easily distinguished from peak-limited complex Gaussian signals, regardless of the parameters used for this simple average-power statistic. On the other hand, it is not always the case that higher-order QAM can be distinguished from peak-limited complex Gaussian signals.

V. Conclusions

This paper presented a low-complexity discrimination statistic to distinguish QAM signals from Gaussian signals in peak-limited AWGN channels. The statistic is an average power estimate that includes K iid samples of the signal. Using theoretical mean and variance values for the power of the received signals, we found the discrimination range for the statistic. In deriving the mean power of a SLM PWLS complex Gaussian signal, we found a closed-form value for $E[1/PAR]$, which was an open research problem. The results show that the statistic is robust for discriminating 4QAM from Gaussian signals.

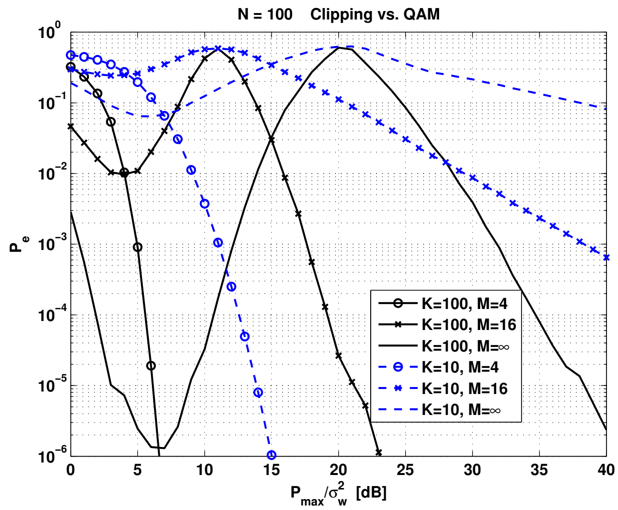

Fig. 4. Probability of error in detecting a QAM signal versus an optimally clipped complex Gaussian signal.

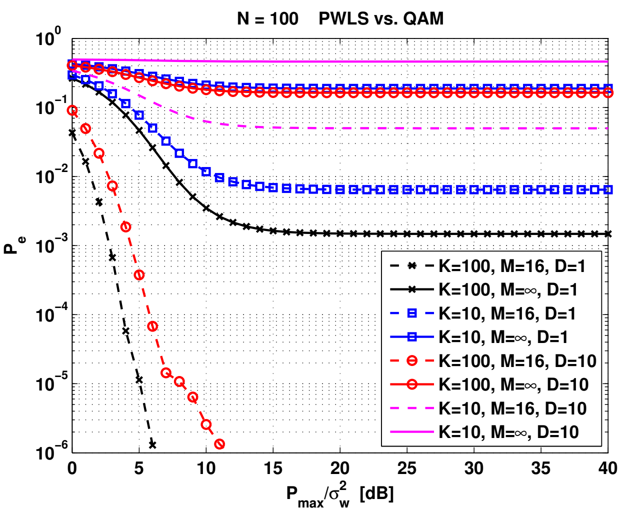

Fig. 5. Probability of error in detecting a QAM signal versus an SLM PWLS complex Gaussian signal.

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APPENDIX

PAR Harmonic Mean: From (11), we can expand the polynomials in the pdf to get

$$E[1/PAR] = ND \sum_{n=0}^{N-1} \sum_{d=0}^{D-1} \sum_{p=0}^{Nd} (-1)^{n+d+p} \binom{N-1}{n}\binom{D-1}{d}\binom{Nd}{p} \int_0^\infty \frac{e^{-(n+p+1)x}}{x} dx. \quad (34)$$

However,

$$\int_0^\infty \frac{e^{-(n+p+1)x}}{x} dx \to \infty \quad (35)$$

which means that (34) can not be calculated by evaluating (35) for every n and p. Instead, note that

$$\begin{aligned} \int_0^\infty \frac{e^{-\alpha x}}{x} dx &= \lim_{x\to 0} \mathrm{E}_1(\alpha x) \\ &= \lim_{x\to 0} -\gamma - \ln(\alpha x) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(\alpha x)^k}{k k!} \\ &= -\gamma - \ln(\alpha) - \lim_{x\to 0} \ln(x), \end{aligned} \quad (36)$$

where $\mathrm{E}_1(x)$ is the exponential integral and γ is the Euler-Mascheroni constant. See [14, p. 127] for details on the series expansion of $\mathrm{E}_1(x)$. Note that the quantity in (36) still approaches ∞. However, when (36) is substituted into (34), the constant, namely, $-\gamma - \lim_{x\to 0} \ln(x)$, even though it is infinitely large, will cancel out. That is

$$ND \sum_{n=0}^{N-1} \sum_{d=0}^{D-1} \sum_{p=0}^{Nd} (-1)^{n+d+p} \binom{N-1}{n}\binom{D-1}{d}\binom{Nd}{p} (-\gamma - \lim_{x\to 0} \ln(x)) = 0, \quad (37)$$

which can be proved by substituting $e^{-x} = 1$ in to (10) and observing that the resulting expression is equal to zero. Thus, the harmonic mean of the PAR is the sum of the only term remaining from (36), $-\ln(\alpha)|_{\alpha=n+p+1}$, and is given in (12).

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